

# Rigidity of formal characters of Lie algebras (II)

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## Abstract

For a complex simple Lie algebra of type  $A_l, B_l, C_l$  or  $D_l$ , given a family of elements  $f_\lambda \in \mathbb{Z}[\Lambda], \lambda \in \Lambda^+$ , we show that  $f_\lambda$  is just the formal character of the Weyl module  $V(\lambda)$  if  $f_\lambda$  satisfy several natural conditions. Hence we give a necessary and sufficient condition for constructing a family of  $\mathfrak{g}_l$ -modules from a family of  $\mathfrak{g}_{l-1}$ -modules.

**Keywords:** formal characters, tensor product, Weyl modules

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## 1 Introduction

Let  $\mathfrak{g}_l$  be a complex simple Lie algebra and  $V(\lambda)$  be the Weyl module. The formal character of  $V(\lambda)$

$$\text{ch}_\lambda = \sum_{x \in \Pi(\lambda)} m_\lambda(x) e(x) = \sum_{\mu \in \Pi^+(\lambda)} m_\lambda(\mu) h(\mu), \quad \lambda \in \Lambda^+$$

is determined by the Weyl character formula or by other methods, such as the Freudenthal formula or Kostant formula. All the formal characters  $\text{ch}_\lambda, \lambda \in \Lambda^+$  is a basis of group ring  $\mathbb{Z}[\Lambda]^W$  which is invariant under action of Weyl group  $W$ . The Weyl module  $V(\lambda)$  is also a  $\mathfrak{g}_{l-1}$ -module for the natural

Lie subalgebra  $\mathfrak{g}_{l-1}$  of Lie algebra  $\mathfrak{g}_l$ ,  $l \geq 2$ . Especially, we have

$$\text{If } \lambda - \mu = \sum_{i=1}^{l-1} k_i \alpha_i + 0\alpha_l, \text{ then } m_\lambda(\mu) = m_{\lambda|_{\mathfrak{g}_{l-1}}}(\mu|_{\mathfrak{g}_{l-1}}). \quad (*)$$

The product of  $\text{ch}_\mu$  and  $\text{ch}_\nu$  is defined by tensor product  $V(\mu) \otimes V(\nu)$  as follow:

$$\text{ch}_\mu \text{ch}_\nu = \sum_{\lambda \in \Lambda^+} c_{\mu, \nu}^\lambda \text{ch}_\lambda.$$

The Littlewood-Richardson coefficient  $c_{\mu, \nu}^\lambda$  defined the multiplicity of  $V(\lambda)$  in  $V(\mu) \otimes V(\nu)$  can be determined according to the formal characters  $\text{ch}_\lambda$ ,  $\text{ch}_\mu$  and  $\text{ch}_\nu$ . According to the complete reducibility of  $\mathfrak{g}_l$ -modules and

$$\dim \text{Hom}(U \otimes V, W) = \dim \text{Hom}(U, W \otimes V^*),$$

then

$$c_{\mu, \nu}^\lambda = c_{\lambda, -w_0\nu}^\mu. \quad (**)$$

In [7] we prove a theorem for Lie algebra  $\mathfrak{g}_l$  of type  $A$  that

*Given a family of elements  $f_\lambda$ 's of  $\mathbb{Z}[\Lambda]^W$ ,  $\lambda \in \Lambda^+$ , if the condition  $(*)$  and  $(**)$  are satisfied, then these  $f_\lambda$ 's are just equal to the formal characters  $\text{ch}_\lambda$ 's.*

This theorem describe the relation between  $V(\lambda)$  as  $\mathfrak{g}_l$ -module and as  $\mathfrak{g}_{l-1}$ -module. That is to say, the first condition  $(*)$  tell us the  $f_\lambda$  describe the  $\mathfrak{g}_{l-1}$ -module structure locally, however the second condition  $(**)$  ensure us to construct a  $\mathfrak{g}_l$ -module globally from the local  $\mathfrak{g}_{l-1}$ -module structure. So the theorem give a necessary and sufficient condition for lifting all these  $\mathfrak{g}_{l-1}$ -modules to  $\mathfrak{g}_l$ -modules. We are especially interested in finding out a similar condition in positive characteristic case. This is also the motivation for the paper [7] and this note.

The formal characters  $f_\lambda$ 's are determined by these natural condition completely. This property is called rigidity of formal characters in [7]. We continue our work in [7] and state a similar rigidity theorem 3.3 for Lie algebra  $\mathfrak{g}_l$  of type  $A_l, B_l, C_l$ ,  $l \geq 2$ , or  $D_l$ ,  $l \geq 4$  in this note with some additional condition.

## 2 notations

From now on let  $\mathfrak{g}_l$  be a complex simple Lie algebra of type  $A_l, B_l, C_l, l \geq 2$ , or  $D_l, l \geq 4$ , and let  $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$  be the set of simple roots. Let  $\Lambda$  be the set of weights, and  $\omega_1, \omega_2, \dots, \omega_l$  the set of fundamental dominant weights. Then the set of dominant weights is denoted by  $\Lambda^+$ . Let  $\text{ch } \lambda, \lambda \in \Lambda^+$ , be the formal character of the Weyl module  $V(\lambda)$ , they form a free  $\mathbb{Z}$ -module of the commutative ring  $\mathbb{Z}[\Lambda]$ , with base  $\{e(\lambda), \lambda \in \Lambda\}$ , and multiplication  $e(\lambda) * e(\mu) = e(\lambda + \mu)$ . Let  $W$  be the Weyl group, action on  $\mathbb{Z}[\Lambda]$  naturally as  $\sigma e(\lambda) = e(\sigma\lambda)$ . Set

$$\mathbb{Z}[\Lambda]^W = \{f \in \mathbb{Z}[\Lambda] \mid wf = f, \ w \in W\}.$$

Let  $W_\lambda$  be the  $W$ -orbit of  $\lambda$  and  $h(\lambda) = \sum_{x \in W_\lambda} e(x)$ . Let  $\Pi(\lambda)$  be the set of saturated weights of weight  $\lambda$  and  $\Pi^+(\lambda) = \Pi(\lambda) \cap \Lambda^+$ . It is well known that

$$\text{ch } \lambda = \sum_{x \in \Pi(\lambda)} m_\lambda(x) e(x) = \sum_{\mu \in \Pi^+(\lambda)} m_\lambda(\mu) h(\mu), \quad \lambda \in \Lambda^+,$$

forms a basis of  $\mathbb{Z}[\Lambda]^W$ .

Recall that  $w_0$  is the longest element of  $W$  and  $c_{\mu, \nu}^\lambda$  is the Littlewood-Richardson coefficient. According to the complete reducibility of  $\mathfrak{g}_l$ -modules and

$$\dim \text{Hom}(U \otimes V, W) = \dim \text{Hom}(U, W \otimes V^*),$$

we have

$$[V(\mu) \otimes V(\lambda) : V(\nu)] = [V(\nu) \otimes V(\lambda)^* : V(\mu)] = [V(\nu) \otimes V(-w_0\lambda) : V(\mu)].$$

Hence

$$c_{\mu, \lambda}^\nu = c_{\nu, -w_0\lambda}^\mu.$$

## 3 main results

**3.1.** Let  $\beta = \sum_{i=1}^l k_i \alpha_i$ , define

$$\text{Supp}(\beta) = \{\alpha_i \mid k_i > 0\}.$$

For  $\lambda \in \Lambda^+$ , set

$$f_\lambda = \sum_{x \in \Pi(\lambda)} n_\lambda(x) e(x) = \sum_{\mu \in \Pi^+(\lambda)} n_\lambda(\mu) h(\mu) \in \mathbb{Z}[\Lambda]^W$$

satisfied  $n_\lambda(\lambda) = 1, n_\lambda(x) = 0$ , if  $x \notin \Pi(\lambda)$ . Then  $f_\lambda$  is also a basis of  $\mathbb{Z}[\Lambda]^W$ . Hence there exists unique  $n_{\mu, \nu}^\lambda$ , such that

$$f_\mu * f_\nu = \sum_{\lambda \in \Pi^+(\mu + \nu)} n_{\mu, \nu}^\lambda f_\lambda.$$

By the definition of  $f'_\lambda s$ , we have

$$n_{\mu, \nu}^{\mu + \nu} = 1; n_{\mu, \nu}^\lambda = n_{\nu, \mu}^\lambda.$$

For dominant weights  $\lambda, t, \mu, \nu$ , the two numbers  $n_\lambda(t)$  and  $n_{\mu, \nu}^t$  is determined by each other in some sense as follow c.f [7] 3.2.

**Lemma 3.1.** *If  $\lambda = \mu + \nu, \mu \neq 0, \nu \neq 0$  then*

$$n_\lambda(t) = n_{\mu, \nu}^t + g(n_\mu(y), n_\nu(z), n_s(x)),$$

where function  $g(n_\mu(y), n_\nu(z), n_s(x))$  is determined by those  $n_\mu(y), n_\nu(z), n_s(x)$ , with  $\mu \not\leq \lambda, \nu \not\leq \lambda, s \leq \lambda$  and  $\mu - y \leq \lambda - t, \nu - z \leq \lambda - t, s - x \not\leq \lambda - t$ .

We list some facts on fundamental dominant weights  $\omega_i$  c.f.[2].

(1) For type  $A_l$ ,

$$\begin{aligned} \omega_1 - w_0\omega_1 &= \omega_1 + \omega_l = \omega_l - w_0\omega_l = \alpha_1 + \alpha_2 + \cdots + \alpha_l; \\ \omega_i - w_0\omega_i &= \omega_i + \omega_{l-i+1} = \alpha_1 + 2\alpha_2 + \sum_{j=3}^l k_j \alpha_j, \quad 1 < i < l. \end{aligned}$$

(2) For type  $B_l$ ,

$$\begin{aligned} \omega_1 - w_0\omega_1 &= 2\omega_1 = 2(\alpha_1 + \alpha_2 + \cdots + \alpha_l); \\ \omega_l - w_0\omega_l &= 2\omega_l = \alpha_1 + 2\alpha_2 + \cdots + l\alpha_l; \\ \omega_i - w_0\omega_i &= 2\omega_i = 2(\alpha_1 + 2\alpha_2) + \sum_{j=3}^l k_j \alpha_j, \quad 1 < i < l. \end{aligned}$$

(3) For type  $C_l$ ,

$$\begin{aligned} \omega_1 - w_0\omega_1 &= 2\omega_1 = 2(\alpha_1 + \alpha_2 + \cdots + \alpha_{l-1}) + \alpha_l; \\ \omega_i - w_0\omega_i &= 2\omega_i = 2(\alpha_1 + 2\alpha_2) + \sum_{j=3}^l k_j \alpha_j, \quad 1 < i. \end{aligned}$$

(4) For type  $D_l$ ,

$$\omega_1 - w_0\omega_1 = 2\omega_1 = 2(\alpha_1 + \alpha_2 + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l;$$

$$\omega_l - w_0\omega_l = \omega_{l-1} + \omega_l = \alpha_1 + 2\alpha_2 + \cdots + (l-2)\alpha_{l-2} + (l-1)(\alpha_{l-1} + \alpha_l);$$

$$\omega_i - w_0\omega_i = \alpha_1 + 2\alpha_2 + \sum_{j=3}^l k_j \alpha_j, \quad 1 < i < l.$$

From the definition of  $\text{Supp}(\beta)$  and the facts on  $\omega_i - w_0\omega_i$  in above, we have

**Lemma 3.2.** *Let  $\mathfrak{g}_l$  be a complex simple Lie algebra of type  $A_l, B_l, C_l$  or  $D_l$  and  $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_l + \sum_{j=1}^l k_j \alpha_j$ . Then the following statements hold:*

1.  $|\text{Supp}(\omega_i - w_0\omega_i - \beta)| < l$ , if  $k_1 \geq 1$  or  $k_2 \geq 2$ .
2.  $|\text{Supp}(\omega_1 - w_0\omega_1 - \beta)| < l$ , for  $A_l, C_l, D_l$ .
3.  $|\text{Supp}(\omega_l - w_0\omega_l - \beta)| < l$ , for  $A_l, B_l, D_l$ .
4.  $|\text{Supp}(\omega_1 - w_0\omega_1 - \beta)| < l$ , if  $\sum_{j=1}^l k_j > 0$  for  $B_l$ .
5.  $|\text{Supp}(\omega_i - w_0\omega_i - \beta)| < l$ , if  $\omega_i$  is the minimal weight.

**3.2.** Let  $\lambda, \mu \in \Lambda^+$ , define  $\mu < \lambda$  if  $\mu \prec \lambda$  or  $\lambda - \mu \in \Lambda^+$ .

Now we state the main theorem in this paper.

**Theorem 3.3.** *Let  $\mathfrak{g}_l$  be a complex simple Lie algebra of type  $A_l, B_l, C_l$ ,  $l \geq 2$ , or  $D_l$ ,  $l \geq 4$ , if these  $f_\lambda$ 's satisfy the following conditions:*

(1)  $n_\lambda(\mu) = m_\lambda(\mu)$ , if  $|\text{Supp}(\lambda - \mu)| < l$ .

(2)  $n_\lambda(\mu) = m_\lambda(\mu)$ , if  $\lambda_1 = 0$ ,  $\lambda - \mu = \alpha_1 + 2\alpha_2 + \sum_{i=3}^l t_i \alpha_i$ ,  $t_i \geq 1$ .

(3)  $n_\lambda(\mu) = m_\lambda(\mu)$ , if  $\lambda_1 \neq 0$ ,  $\lambda - \mu = \alpha_1 + \alpha_2 + \cdots + \alpha_l$  for type  $B_l$ .

(4)  $n_{\mu, \nu}^\lambda = n_{\lambda, -w_0\nu}^\mu$  for  $\lambda, \mu, \nu \in \Lambda^+$ .

Then  $f_\lambda = ch_\lambda$ ,  $n_{\mu, \nu}^\lambda = c_{\mu, \nu}^\lambda$ .

*Proof.* It is only need to prove  $n_\lambda(\mu) = m_\lambda(\mu)$ ,  $\lambda \in \Lambda^+$ ,  $\mu \in \Pi^+(\lambda)$ . We will prove the theorem by induction on  $\Lambda^+$  with the partial order “ $<$ ” and on  $\Pi(\lambda)$  with the partial order “ $\prec$ ”.

Firstly, if  $\lambda = \omega_i$  or 0 is a minimal weight, then their saturated weight set  $\Pi(\lambda)$  only contains one dominate weight. So the theorem holds by the definition of  $f_\lambda$ .

Suppose that  $\lambda \in \Lambda^+$  not be a minimal weight. Let  $\mu \in \Pi^+(\lambda)$ ,  $\beta = \lambda - \mu$ . We will consider the different cases as follows:

(1) When  $|\text{Supp}(\beta)| < l$ , by the first condition in theorem then

$$n_\lambda(\mu) = m_\lambda(\mu).$$

(2) When  $|\text{Supp}(\beta)| = l$  and  $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_l + \sum_{j=1}^l k_j \alpha_j$ .

Because  $\lambda \neq 0$ , there exists  $i$  such that  $\lambda_i \neq 0$ . So  $\lambda - \omega_i \in \Lambda^+$ . By the fourth condition then

$$n_{\lambda - \omega_i, \omega_i}^\mu = n_{\mu, -w_0 \omega_i}^{\lambda - \omega_i}.$$

Noticing that

$$\mu + (-w_0 \omega_i) - (\lambda - \omega_i) = \omega_i - w_0 \omega_i - \beta.$$

(i) When  $k_1 \geq 1$  or  $k_2 \geq 2$ , by lemma 3.2,  $|\text{Supp}(\omega_i - w_0 \omega_i - \beta)| < l$ . Then by the first condition in the theorem

$$n_{\mu + (-w_0 \omega_i)}(\lambda - \omega_i) = m_{\mu + (-w_0 \omega_i)}(\lambda - \omega_i).$$

Moreover we have

$$\begin{aligned} n_\mu(y) &= m_\mu(y), & \text{if } \mu - y \preceq \omega_i - w_0 \omega_i - \beta; \\ n_{-w_0 \omega_i}(z) &= m_{-w_0 \omega_i}(z), & \text{if } -w_0 \omega_i - z \preceq \omega_i - w_0 \omega_i - \beta; \\ n_s(x) &= m_s(x) & \text{if } s - x \preceq \omega_i - w_0 \omega_i - \beta. \end{aligned}$$

Then by lemma 3.1

$$\begin{aligned} n_{\mu, -w_0 \omega_i}^{\lambda - \omega_i} &= n_{\mu + (-w_0 \omega_i)}(\lambda - \omega_i) - g(n_\mu(y), n_{-w_0 \omega_i}(z), n_s(x)) \\ &= m_{\mu + (-w_0 \omega_i)}(\lambda - \omega_i) - g(m_\mu(y), m_{-w_0 \omega_i}(z), m_s(x)) \\ &= c_{\mu, -w_0 \omega_i}^{\lambda - \omega_i}. \end{aligned}$$

Hence by the fourth condition in the theorem and the property of  $c_{\mu,\nu}^\lambda$  in (\*\*)

$$n_{\lambda-\omega_i, \omega_i}^\mu = n_{\mu, -w_0\omega_i}^{\lambda-\omega_i} = c_{\mu, -w_0\omega_i}^{\lambda-\omega_i} = c_{\lambda-\omega_i, \omega_i}^\mu.$$

By lemma 3.1 and induction hypothesis again, we have

$$n_\lambda(\mu) = m_\lambda(\mu).$$

(ii) When  $k_1 = 0$  and  $k_2 = 1$  or  $k_1 = 0$  and  $k_2 = 0$

If  $\lambda_1 = 0$ , by the second condition in theorem then

$$n_\lambda(\mu) = m_\lambda(\mu)$$

in the first case. However we have  $\lambda_1 \neq 0$  in the second case because

$$\lambda = \mu + \alpha_1 + \alpha_2 + \sum_{i=3}^l k_i \alpha_i = \mu + (1, \dots).$$

If  $\lambda_1 \neq 0$ , choose  $\lambda_i = \lambda_1$ , then

$$n_{\lambda-\omega_1, \omega_1}^\mu = n_{\mu, -w_0\omega_1}^{\lambda-\omega_1}.$$

For

$$\mu + (-w_0\omega_1) - (\lambda - \omega_1) = \omega_1 - w_0\omega_1 - \beta,$$

by lemma 3.2,  $|\text{Supp}(\omega_1 - w_0\omega_1 - \beta)| < l$  for Lie algebra  $\mathfrak{g}_l$  of type  $A_l, C_l, D_l$ ; type  $B_l$  in the first case, and type  $B_l$  in the second case with additional condition  $\sum_{j=1}^l k_j \alpha_j > 0$ , so

$$n_\lambda(\mu) = m_\lambda(\mu)$$

by lemma 3.1 and induction hypothesis as we prove in the case (i).

The above equation also holds for type  $B_l$  in the second case when  $\sum_{j=1}^l k_j \alpha_j = 0$  by the third condition in the theorem.

We complete the proof of theorem for Lie algebra  $\mathfrak{g}_l$  of type  $A_l, B_l, C_l, D_l$ .

**3.3 Remarks.** 1. As we mentioned in the introduction this theorem describe when we can construct a family of  $\mathfrak{g}_l$ -modules from a family of  $\mathfrak{g}_{l-1}$ -modules. However it need more conditions for type  $B_l, C_l, D_l$  than type  $A_l$  in[7]. This is because there exist some fundamental dominant weights which are no longer minimal weight. It need also these conditions to obtain a similar theorem for Lie algebra of type  $E_l, F_4$  and  $G_2$ .

2. The number  $m_\lambda(\mu)$  in the second and third condition can be determined precisely, c.f. theorem 4 in [4], theorem 3.10 in [5] and theorem 4.9 in [6].

3. The first condition is very natural and the fourth condition can be weakened. In our proof it need only  $n_{\mu,\nu}^\lambda = n_{\lambda,-w_0\nu}^\mu$  for  $\lambda, \mu \in \Lambda^+$  and  $\nu = \omega_i$  be a fundamental dominant weight. We will investigate and generalize the weaker condition in positive characteristic in the future.

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